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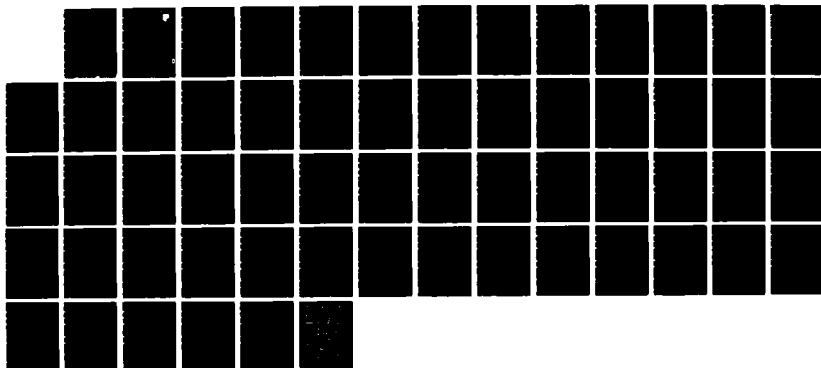
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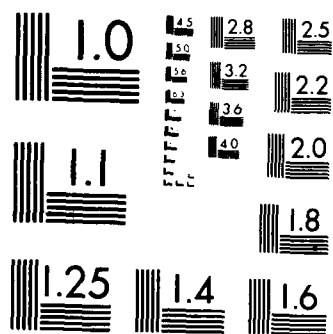
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Interim Report
December 1987

STOCHASTIC MODELING OF EM SCATTERING FROM FOLIAGE

Systems Engineering, Inc.

Gilmore L. Blankenship

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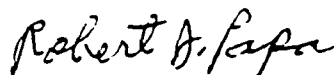


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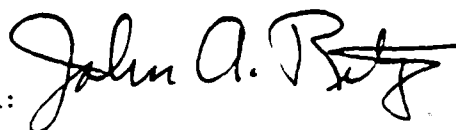
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1 Introduction and Summary

During the first year of this program, we focussed on development of analytical methods for representation of multiple scattering effects in heterogeneous media. We have examined several methods for representation and computation of effective parameter models for the media in terms of statistical characterizations of the media. These methods lead to representation for the effective parameters, e.g., the effective dielectric constant or conductivity in a heterogeneous medium. Since these representations may be difficult to evaluate in specific cases, we have also examined current research on the derivation of *bounds* for the effective parameters as a means of quickly providing good approximations to the behavior of the medium.

Our work on the T -matrix methods is summarized in section 1. We plan to continue this work to develop numerical algorithms for the evaluation of scattering cross sections and absorption/dissipation mechanisms in typical foliage configurations. It is important to emphasize that the T -matrix methods, though formal in structure, provide simple models for multiple scattering phenomena which are valid for a wide range of frequencies (of the incident radiation).

The work on the use of "multiple-scale" asymptotic analysis (homogenization) for the representation of scattering processes is given in section 2. This is primarily a "low frequency" theory, valid when the wave length of the incident radiation exceeds the characteristic dimensions of inter-

particle separation in the scattering medium. The homogenization method does, however, provide a systematic procedure for the construction of high order approximations to the "effective parameter" representations of the scattering process. These approximations may be evaluated by common numerical procedures when the scattering medium has a periodic structure. We describe this case first in section 3, before turning to the more realistic, but less tractable, case when the physical model parameters (permittivity, conductivity, and permeability) are random functions of position in the medium. We have not completed our analysis of the probabilistic case at this writing, and we report only on a basic mathematical result for a key quantity arising in the homogenization method.

In section 4 we examine the derivation and use of "bounds" on the effective parameters in a scattering problem. The derivation of bounds is an old subject in scattering theory, dating from early in this century. Nevertheless, there has been some very interesting recent work on various methods for deriving bounds. We discuss this work briefly in section 4, as it applies to the problem of scattering from foliage. As with homogenization, the class of bounds discussed applies mostly to the low frequency case.

2 T-matrix Approximations to Multiple Scattering

In this section we consider a classical approach based on the "T-matrix formalism" for the problem of modeling scattering in a region containing two or more types of dielectric scatterers arranged in a random geometry in a surrounding medium. Models for scattering from a medium containing a two (or more) classes of (randomly oriented) scatterers in an enveloping medium are prescribed in terms of the definition and derivation of approximations for the "effective dielectric constant" resulting in a family of approximations for an "effective scattering representation" for heterogenous media.

2.1 Basic Framework

Consider a region O of space occupied by a homogeneous (background) dielectric medium with dielectric constant ϵ_0 containing elements from two classes of scatterers with dielectric constants ϵ_1, ϵ_2 . Let $v_{ij}, i = 1, 2, j = 1, 2, \dots, N_i$, be the subsets of O occupied by scattering elements j of class i . The dielectric properties of the composite material in O are described by

$$\epsilon(x) = \epsilon_0 + \sum_{i=1}^2 (\epsilon_i - \epsilon_0) \sum_{j=1}^{N_i} \chi_{v_{ij}}(x) \quad (1)$$

where

$$\chi_{v_{ij}}(x) = \begin{cases} 1 & x \in v_{ij} \\ 0 & x \notin v_{ij} \end{cases} \quad (2)$$

Let δ_i be a (dimensionless) parameter describing a characteristic dimension of elementary scatterers in class i (e.g., the radius of spherical scatterers); and let ρ_i be the total volume fraction of O occupied by scatterers of class $i = 1, 2$. Suppose a constant (in amplitude - time harmonic) field is incident on the region O .

We wish to characterize the scattering properties of the composite material in the limit $N_i \rightarrow \infty$, $\delta_i \rightarrow 0$, $i = 1, 2$, with ρ_1, ρ_2 constant. We are particularly interested in evaluating the relative roles of the two classes of scatterers in the scattering process, e.g., in terms of their relative densities ρ_i in the cases (a) $\rho_1, \rho_2 \rightarrow 0$ with $\rho_i/\rho_j \sim O(1)$ and (b) $\rho_1, \rho_2 \rightarrow 0$ with $\rho_i/\rho_j \rightarrow 0$.

We shall sketch the development of a family of effective scattering approximations for the material using the multiple scattering framework developed originally by M. Lax (1951,1952).

2.2 Transition Operator Representations

We use the *transition operator* formalism of classical scattering theory (Lax 1951,1952,1973) as used by (Lang 1981) and (Kohler and Papanicolaou 1981) as a framework for the derivation of several formulas for an *effective*

dielectric constant for the composite medium. Maxwell's equations for this situation may be written in the form

$$\begin{aligned}\nabla \cdot (\epsilon_0 E) + \sum_{i=1}^2 (\epsilon_i - \epsilon_0) \sum_{j=1}^{N_i} \nabla \cdot [\chi_{v_{ij}} E] &= 0 \\ \nabla \times E &= 0\end{aligned}\quad (3)$$

with the boundary condition

$$\langle E \rangle \rightarrow \bar{E} \quad \text{as } N_1, N_2 \rightarrow \infty \quad (4)$$

where $\langle \cdot \rangle$ is expectation and \bar{E} is the constant external field. The constituent relationship

$$D(x) = \epsilon(x)E(x) \quad (5)$$

and the condition

$$\langle D(\cdot) \rangle = \epsilon^* \bar{E} \quad (6)$$

defines the *effective dielectric constant* for the composite material.

Rewriting (3) in abstract form

$$(L_0 + M)E = 0, \quad \nabla \times E = 0 \quad (7)$$

where

$$\begin{aligned}L_0 &= \nabla \cdot (\epsilon_0 \cdot) \\ M &= M_1 + M_2, \quad M_i = \sum_{j=1}^{N_i} V_{ij}\end{aligned}\quad (8)$$

$$V_{ij} = (\epsilon_i - \epsilon_0) \cdot \nabla [\chi_{v,ij}]$$

we have

$$E + L_0^{-1} M E = F \quad (9)$$

as the integral form of (3). F is chosen so that

$$L_0 F = 0, \quad \langle E \rangle = \bar{E}. \quad (10)$$

The *transition operator* ("matrix") T is defined as follows:

$$\begin{aligned} L_0^{-1} M &= (L_0 + M - M)^{-1} M \\ &= [(L_0 + M)(I - (L_0 + M)^{-1} M)]^{-1} M \\ &= T(I - T)^{-1} \end{aligned} \quad (11)$$

with $T = (L_0 + M)^{-1} M$. So (9) becomes

$$[I + T(I - T)^{-1}] E = F \quad (12)$$

or

$$E = (I - T) F \quad (13)$$

The condition (10) gives

$$F = (I - \langle T \rangle)^{-1} \bar{E} \quad (14)$$

$$E = (I - T)(I - \langle T \rangle)^{-1} \bar{E} \quad (15)$$

Using (5)(6)(15), we have

$$\epsilon \cdot \bar{E} = (\langle \epsilon \rangle - \langle \epsilon T \rangle)(I - \langle T \rangle)^{-1} \bar{E} \quad (16)$$

as the definition of ϵ^* in terms of the transition operator. It is apparent that knowledge of T completely characterizes the scattering properties of the composite medium. The operator T is related to the scattering amplitudes of the individual particles, and to their interactions. For simple (single) dipole scatterers, T can be determined from the polarizability of the element; and so, T is defined in terms of quantities of physical interest. Other properties of the transition operator are described in (Lax 1951,1952,1973), (Lang 1981), and (Kohler and Papanicolaou 1981). Our analysis of the representation (15)(16) is based on (Kohler and Papanicolaou 1981).

To bring out the roles of the two classes of scatterers, we reconsider (9)

$$E + L_0^{-1}(M_1 + M_2)E = F \quad (17)$$

which may be written as

$$E = F - T_1 E_1 - T_2 E_2 \quad (18)$$

with

$$\begin{aligned} L_0^{-1}M_i &= T_i(I - T_i)^{-1} \\ T_i &= (L_0 + M_i)^{-1}M_i \\ E_i &= (I - T_i)^{-1}E \end{aligned} \quad (19)$$

since $E_i - T_i E_i = E$, we have

$$E_i + \sum_{k \neq i}^2 T_k E_k = F \quad (20)$$

(The sum in (20) is degenerate; but this is the appropriate form when O contains $K > 2$ classes of scatterers.) Equations (18)-(20) express the field in terms of the fields and transition operators associated with each *class* of scatterers in the composite medium.

Now

$$\begin{aligned} T &= (L_0 + M_1 + M_2)^{-1}(M_1 + M_2) \\ &= (I + T_{12})^{-1}T_1 + (I + T_{21})^{-1}T_2 \end{aligned} \quad (21)$$

where the pair transition operators are

$$T_{ij} = (L_0 + M_i)^{-1}M_j, \quad , i \neq j = 1, 2. \quad (22)$$

This representation gives the formula

$$\begin{aligned} \epsilon^* \sim [< \epsilon > - < \epsilon(I + T_{12})^{-1}T_1 > - < \epsilon(I + T_{12})^{-1}T_2 >] \\ \cdot [I - < (I + T_{12})^{-1}T_1 > - < (I + T_{12})^{-1}T_2 >]^{-1} \end{aligned} \quad (23)$$

for the effective dielectric constant in which the roles of each class of scatterers and their interaction is explicit. We could continue this process to make the role of each elementary scatterer of class i explicit. Note that (23) contains no approximations.

2.3 Average T-Matrix Approximation (ATA)

Expanding

$$T = T_1 + T_2 - T_{12}T_1 - T_{21}T_2 + \dots \quad (24)$$

neglecting the higher order terms ($T_{12}T_1, \dots$), and substituting in (23), we obtain the simple approximation

$$\epsilon^* \sim \langle \epsilon \rangle - \sum_{j=1}^2 \langle \epsilon T_j \rangle \left| I - \sum_{i=1}^2 \langle T_i \rangle \right|^{-1} \quad (25)$$

called the *average T-matrix approximation* (ATA) (Lax 1951,1973) (Kohler and Papanicolaou 1981). The expressions $\langle T_j \rangle$ and $\langle \epsilon T_j \rangle$ in (25) diverge in typical cases, so the ATA must be used with care.

It is possible to evaluate (25) in the limit as the sizes of the elementary scatterers approaches zero. Suppose each class consists of spherical scatterers having radii $\delta_i, i = 1, 2$ with

$$\rho_i = \frac{4}{3} \pi \delta_i^3 c_i \quad (26)$$

the volume fraction occupied by each class. Here c_i is the average number of sphere centers per unit volume. this case leads to the approximation

$$\epsilon^* \sim \frac{\langle \epsilon \rangle + \rho_1 \epsilon_1 \left[\frac{\epsilon_0 - \epsilon_1}{2\epsilon_0 + \epsilon_1} \right] \rho_2 \epsilon_2 \left[\frac{\epsilon_0 - \epsilon_2}{2\epsilon_0 + \epsilon_2} \right]}{1 + \rho_1 \epsilon_1 \left[\frac{\epsilon_0 - \epsilon_1}{2\epsilon_0 + \epsilon_1} \right] \rho_2 \epsilon_2 \left[\frac{\epsilon_0 - \epsilon_2}{2\epsilon_0 + \epsilon_2} \right]} \quad (27)$$

which is a version of the *Claussius - Mossotti* formula (Kohler and Papanicolaou 1981, p. 213). In general, the *Claussius - Mossotti* formula is viewed as a good approximation for low to moderate volume fractions $\rho_i, i = 1, 2$.

The ATA not only contains divergent terms as a rule, it also fails to account for the interactions between scatterers. A family of more refined approximations has been developed to include interaction effects in formulas for effective scattering representations which are nonetheless computationally feasible (Lax 1951,1952,1973) (Elliott et al 1974).

2.4 Coherent Potential Approximation

The simplest of these is the *coherent potential approximation* (CPA) which results (in effect) from neglecting the difference between the field exciting the medium and the average field. Let β_1, β_2 be parameters to be selected, and consider Maxwell's equations rewritten as

$$\begin{aligned} \sum_{i=1}^2 \nabla \cdot [\beta_i \epsilon_0 E] + \sum_{i=1}^2 \sum_{j=1}^{N_i} \nabla \cdot [(\epsilon_i - \epsilon_0) \chi_{ij} E] \\ + \sum_{i=1}^2 \sum_{j=1}^{N_i} \frac{\epsilon_0}{N_i} (1 - \beta_i) \nabla \cdot E \\ \nabla \times E = 0 \end{aligned} \quad (28)$$

We may write this abstractly as

$$(L^\beta + M^\beta)E = 0 \quad (29)$$

with

$$\begin{aligned} L^\beta &= \nabla \cdot (\beta_1 \epsilon_0 \cdot) + \nabla \cdot (\beta_2 \epsilon_0 \cdot) \\ M^\beta &= \sum_{i=1}^2 \sum_{j=1}^{N_i} V_{ij}^\beta \end{aligned} \quad (30)$$

$$V_{ij}^\beta = \nabla \cdot [(\epsilon_i - \epsilon_0) \chi_{ij} \cdot] + \frac{\epsilon_0}{N_i} (1 - \beta_i) \nabla \cdot E$$

The same argument as before yields the representation

$$\epsilon^* \sim \langle \epsilon \rangle - \sum_{i=1}^2 \sum_{j=1}^{N_i} \langle \epsilon T_{ij}^\beta \rangle [1 - \sum_{i=1}^2 \sum_{j=1}^{N_i} \langle T_{ij}^\beta \rangle]^{-1} \quad (31)$$

where

$$T_{ij}^\beta = [L_i^\beta + V_{ij}^\beta]^{-1} V_{ij}^\beta \quad (32)$$

The CPA is based on choosing β to "optimize" the approximation in (31).

Let

$$w^{ij}(x) = (T_{ij}^\beta g)(x) \quad (33)$$

so

$$\begin{aligned} \nabla \cdot [\beta_i \epsilon_0 w^{ij}] + \nabla \cdot [(\epsilon_i - \epsilon_0) \chi_{v_{ij}} w^{ij} + \frac{\epsilon_0}{N_i} (1 - \beta_i) w^{ij}] \\ = \nabla \cdot [(\epsilon_i - \epsilon_0) \chi_{v_{ij}} g + \frac{\epsilon_0}{N_i} (1 - \beta_i) g] \end{aligned} \quad (34)$$

$$\nabla \times w^{ij} = 0$$

Let

$$\epsilon_i^{\beta, N} = [\beta_i \epsilon_0 + \frac{1}{N_i} (1 - \beta_i)] \epsilon_0 \approx \beta_i \epsilon_0 \quad \text{for } N_i \text{ large} \quad (35)$$

and $w^{ij} = w_1^{ij} + w_2^{ij}$ where

$$\nabla \cdot (\epsilon_i^{\beta, N} w_1^{ij}) + \nabla \cdot [(\epsilon_i - \epsilon_0) \chi_{v_{ij}} w_1^{ij}] \quad (36)$$

$$= \nabla \cdot [(\epsilon_i - \epsilon_0) \chi_{v_{ij}} g]$$

$$\nabla \cdot (\epsilon_i^{\beta, N} w_2^{ij}) + \nabla \cdot [(\epsilon_i - \epsilon_0) \chi_{v_{ij}} w_2^{ij}] \quad (37)$$

$$= \frac{\epsilon_0}{N_i} (1 - \beta_i) \nabla \cdot g$$

$$\nabla \times w_1^{ij} = 0 = \nabla \times w_2^{ij}$$

Let $\{y_{ij}^N\}$ be the locations of the centers of the individual scattering elements. For δ_i small, we may replace g on the right in (36) by $g(y_{ij}^N) = g_{ij}$.

Using this and the approximation

$$\epsilon_i^{\beta, N} \approx \beta_i \epsilon_0 \quad \text{for } N_i \text{ large} \quad (38)$$

we have

$$w_1^{ij}(x) \sim -\frac{\epsilon_0 - \epsilon_i}{3\beta_i\epsilon_0 + \epsilon_i - \epsilon_0} \delta_i^3 \nabla_z \left[\frac{g_{ij} \cdot (x - y_{ij}^N)}{|x - y_{ij}^N|^3} \right], |x - y_{ij}^N| > \delta_i \quad (39)$$

$$w_1^{ij}(x) \sim -\frac{\epsilon_0 - \epsilon_i}{3\beta_i\epsilon_0 + \epsilon_i - \epsilon_0} g_{ij}, |x - y_{ij}^N| < \delta_i$$

a result similar to (Kohler and Papanicolaou 1981, sect. 8). In addition, using (38) in (37)

$$w_2^{ij} = \frac{(1 - \beta_i)}{4\pi\beta_i N_i} \nabla_z \int \frac{(x - y) \cdot g(y)}{|x - y|^3} dy + w_3^{ij} \quad (40)$$

where

$$w_3^{ij}(x) \sim +\frac{\epsilon_0 - \epsilon_i}{3\beta_i\epsilon_0 + \epsilon_i - \epsilon_0} \frac{(1 - \beta_i)}{N_i} \delta_i^3 \nabla_z \left[\frac{g_{ij} \cdot (x - y_{ij}^N)}{|x - y_{ij}^N|^3} \right], |x - y_{ij}^N| > \delta_i \quad (41)$$

$$w_3^{ij}(x) \sim +\frac{\epsilon_0 - \epsilon_i}{3\beta_i\epsilon_0 + \epsilon_i - \epsilon_0} \frac{(1 - \beta_i)}{N_i} g_{ij}, |x - y_{ij}^N| < \delta_i$$

Combining these, we have

$$\begin{aligned} \sum_{i=1}^2 \sum_{j=1}^{N_i} (T_{ij}^\beta g)(x) &\sim \sum_{i=1}^2 \frac{(1 - \beta_i)}{4\pi\beta_i} \nabla_z \int \frac{(x - y) \cdot g(y)}{|x - y|^3} dy \\ &- \sum_{i=1}^2 \frac{\epsilon_0 - \epsilon_i}{3\beta_i\epsilon_0 + \epsilon_i - \epsilon_0} \delta_i^2 \sum_{j=1}^{N_i} (1 - \chi_{v_{ij}}(x)) \nabla_z \left[\frac{g_{ij} \cdot (x - y_{ij}^N)}{|x - y_{ij}^N|^3} \right] \\ &- \sum_{i=1}^2 \frac{\epsilon_0 - \epsilon_i}{3\beta_i\epsilon_0 + \epsilon_i - \epsilon_0} \sum_{j=1}^{N_i} \chi_{v_{ij}}(x) g_{ij} + \sum_{i=1}^2 \sum_{j=1}^{N_i} w_3^{ij}(x) \end{aligned} \quad (42)$$

Arguing as in (Kohler and Papanicolaou 1981, sect. 8), the last term on the right in (42) is $O(N_i^{-1})$ on the average as $N_i \rightarrow \infty$. We take the expectation and choose β so that

$$\left\langle \sum_{i=1}^2 \sum_{j=1}^{N_i} T_{ij}^\beta g \right\rangle = 0 \quad (43)$$

This is the CPA. This requires

$$\frac{(1 - \beta_i)}{3\beta_i} = \rho_i \frac{\epsilon_0 - \epsilon_i}{3\beta_i \epsilon_0 + \epsilon_i - \epsilon_0}, i = 1, 2 \quad (44)$$

where $\rho_i = \frac{4}{3}\pi\delta_i^3 c_i$ with c_i the average number of scattering centers (of class i) per unit volume.

With β_i chosen from (44), we have $\langle \sum_{i=1}^2 \sum_{j=1}^{N_i} T_{ij}^\beta \rangle = 0$, and so, the effective dielectric constant may be approximated using the CPA by evaluating (31).

3 Homogenization and Multiple Scattering

In this section we turn our attention to an alternative method for representation of multiple scattering effects in heterogeneous media. The method is a variation of the "homogenization" procedure which has been used widely in mathematical physics and engineering to develop effective media approximations. This method has promise for several reasons:

1. It leads more naturally than the T -operator formalism to a compact representation for an effective parameter (complex dielectric constant) representation for the effective medium approximation.
2. The resulting representation for the effective dielectric constant includes the interaction of microscopic effects (multiple scattering) explicitly.

3. The underlying analysis applies (however, with significant differences in detail) to both periodic and random media. In the former case it is possible to solve the equations for the approximation; in the latter it is necessary to develop approximations which use physically measurable quantities (second order statistics and correlation functions).
4. One can prove convergence of the scaled model to the "homogenized" model in both the periodic and random case.
5. The representation provides a (formal) basis for the systematic construction of a sequential approximation to the effective dielectric constant, including "higher order" expansions for the effective parameters in terms of the small parameter, at least in the periodic case.

We wish to undertake a systematic investigation of such approximations in the context of scattering from foliage covered terrain, and the multiscale (homogenization) method offers a more general setting for such a comparison than does the T -matrix formalism, at least in the frequency range to which it applies.

3.1 A General Model

To illustrate the ideas, consider the following general model: Let $O \subset \mathbb{R}^3$ be a region in which $\epsilon(x)$, $\mu(x)$ and $\sigma(x)$ are, respectively, the dielectric tensor, magnetic permeability tensor, and conductivity tensor (3×3 matrices) of

the material in region O . In O the (vector-valued) electric and magnetic fields satisfy

$$\frac{\partial}{\partial t} \begin{bmatrix} \epsilon(x) & 0 \\ 0 & \mu(x) \end{bmatrix} \begin{bmatrix} E(x) \\ H(x) \end{bmatrix} = \begin{bmatrix} 0 & \nabla \times \\ -\nabla \times & 0 \end{bmatrix} \begin{bmatrix} E(t, x) \\ H(t, x) \end{bmatrix} \quad (45)$$

$$+ \begin{bmatrix} \sigma(x) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E(t, x) \\ H(t, x) \end{bmatrix}$$

$$E(0, x) = E_0(x), H(0, x) = H_0(x) \quad (46)$$

$$\nabla \cdot [\epsilon E_0] = 0, \nabla \cdot [\mu H_0] = 0 \quad (47)$$

where $\nabla \times$ is the curl operator and $\nabla \cdot = \text{div}$. Note that (47) implies

$$\nabla \cdot [\epsilon E(t, x)] = 0, \nabla \cdot [\mu H(t, x)] = 0, \forall t \geq 0$$

It is necessary to assume that $\epsilon(x), \mu(x)$ and $\sigma(x)$ are symmetric, and that $\epsilon(x), \mu(x)$ are positive definite matrices.

Using

$$\nabla \times E = \begin{bmatrix} 0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\ -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \quad (48)$$

we see that (45) may be written in the abstract form

$$A^0(x) \frac{\partial v}{\partial t} = \sum_{p=1}^3 A^p(x) \frac{\partial v}{\partial x_p} + B(x)v(t, x) \quad (49)$$

where

$$v(t, x) = \begin{bmatrix} E(t, x) \\ H(t, x) \end{bmatrix} \in \mathbb{R}^6 \quad (50)$$

$$A^0(x) = \begin{bmatrix} \epsilon(x) & 0 \\ 0 & \mu(x) \end{bmatrix}$$

the $A^p(t, x)$ are skew-symmetric, 6×6 matrices with 0's and 1's as elements, and

$$B(x) = \begin{bmatrix} \sigma(x) & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{6 \times 6}$$

The symmetry assumption means that (49) is a *symmetric - hyperbolic system* of a type common in mathematical physics. The (asymptotic) analysis of systems of this type with *periodic* coefficients depending on a small (dimensionless) parameter, e.g.,

$$A_\nu^0(x/\nu), A_\nu^p(x/\nu), B_\nu(x/\nu)$$

(for $0 < \nu \ll 1$) was carried out in (Bensoussan, Lions, and Papanicolaou 1978) and in (Sanchez-Palencia 1980) using multi-scale methods (homogenization). The periodic case represents media with regularly imbedded components.

The "inhomogeneities" in the media which govern the scattering of EM radiation from foliage covered terrain are randomly distributed, and so, the homogenization (multi-scale) method must be extended to treat this case. A general theory for partial differential equations with random coefficients was developed in (Papanicolaou and Varadhan 1979). This work forms the basis for our treatment of the random scattering (and absorption) problem.

3.2 Scaling the Model

To set up the analysis, we must prescribe a scaling for the parameters of the random medium, in this case the permittivity, permeability, and conductivity. The general form is

$$\begin{aligned}\epsilon(x) &= \epsilon^\nu(x) = \epsilon(x, x/\nu) \\ \mu(x) &= \mu^\nu(x) = \mu(x, x/\nu) \\ \sigma(x) &= \sigma^\nu(x) = \sigma(x, x/\nu)\end{aligned}\tag{51}$$

In this representation, $\nu > 0$ is a small ($\ll 1$) dimensionless parameter which characterizes the relative scale on which microscopic variations in the medium take place. In previous reports we have shown how this parameter may be identified in terms of the mean free path between elementary scatterers in a foliage covered area.¹

The presence of two scales in (51) suggests that we are permitting variations of the parameters across the structure of two different spatial types. The "fast scale" $y = x/\nu$ captures the effects of microscopic variations; that is, as we pass from one elementary scatterer to the next. Variations on the "slow scale" x capture the effects of macroscopic variations; that is, variations as we move from one type of vegetation to another, or from one region of a given type to another. Note that as y changes by one unit,

¹That is, ν may be defined in terms of the mean free path between scattering centers normalised by a characteristic length of the interaction process, e.g., the length of a scattering region or the principal axis of the first Fresnel zone, etc.

the macroscopic scale x changes by $\nu \ll 1$ units; and as x changes by one unit, y changes by $1/\nu \gg 1$ units.

Using this scaling, there are two basic cases which can be treated by versions of the method of multiple scales:

1. The functions $\epsilon(x, y), \mu(x, y), \sigma(x, y)$ are periodic in their second argument.
2. The functions $\epsilon(x, y), \mu(x, y), \sigma(x, y)$ are random functions of their second argument which are stationary and satisfy a mixing condition.

The first case was treated in (Bensoussan, Lions, and Papanicolaou 1978) and in (Sanchez-Palencia 1980), (see also the work of Tartar and Murat (1986)) among many other related problems. The second case provides a basis for modeling certain types of foliage covered terrain.

The mixing which must be imposed in the second case is that the statistical correlations between random events in the medium must decrease rapidly over the volume of interest. That is, the correlation between values of $\epsilon(x, y_1)$ and $\epsilon(x, y_2)$ decreases as $|y_1 - y_2| \rightarrow \infty$. In effect, this means that the microstructure of the foliage at one point y_1 cannot depend too strongly on that at the (distant) point y_2 . This is clearly reasonable in many situations. It may not, however, be the case in wind blown grass.

Notice that the scattering medium is still permitted to have (statistical

or deterministic) variations which take place in the "macroscopic" x spatial scale.

3.3 The Multiple Scale Hypothesis

The *objective* of the multiple scale asymptotic analysis is to compute approximations to the fields $E^\nu(t, x), H^\nu(t, x)$ in the limit as $\nu \rightarrow 0$, that is, in the limit as the microscopic variations become increasingly dense.

A *formal procedure* for computing approximations is to assume an expansion of the fields in the form

$$E^\nu(x, x/\nu) = E_0(x) + \nu E_1(x, y) + \nu^2 E_2(x, y) + \dots \quad (52)$$

with $y = x/\nu$ and a similar expansion for H^ν . Introducing the change of coordinates

$$\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x} + \frac{1}{\nu} \frac{\partial}{\partial y} \quad (53)$$

in Maxwell's equations, substituting the expansions for E^ν and H^ν , one finds a sequence of equations for the terms $E_k(x, y), k = 0, 1, 2, \dots$ (and H_k). These may be solved using a device called a "corrector" which is derived from a kind of "separation of variables" argument.

If the dissipation is zero ($\sigma = 0$), the result is a representation of the solution for the approximate field in the form

$$\frac{\partial}{\partial t} \begin{bmatrix} q_e(x) & 0 \\ 0 & q_\mu(x) \end{bmatrix} \begin{bmatrix} E_0(x) \\ H_0(x) \end{bmatrix} = \begin{bmatrix} 0 & \nabla \times \\ -\nabla \times & 0 \end{bmatrix} \begin{bmatrix} E_0(t, x) \\ H_0(t, x) \end{bmatrix} \quad (54)$$

$$E_0(0, x) = E_0(x), H_0(0, x) = H_0(x)$$

$$\nabla \cdot [q_\epsilon E_0] = 0, \nabla \cdot [q_\mu H_0] = 0$$

The (matrix-valued) functions $q_\epsilon(x), q_\mu(x)$, which depend on the macroscopic spatial scale, are defined in terms of $\epsilon(x, y), \mu(x, y)$ by the "corrected averages"

$$q_\alpha^{ij}(x) = \mathcal{M} \left[\alpha_{ij} \left(\delta_{ij} + \frac{\partial \chi_j^\alpha(y)}{\partial y_i} \right) \right], i, j = 1, 2, 3 \quad (55)$$

where α_{ij} is either ϵ_{ij} or μ_{ij} , and the function $\chi^\alpha(y)$ is the *corrector* associated with ϵ or μ . Note that the *effective parameters* in the approximation q_ϵ, q_μ are not just the averages of the rapid variations of the parameters $\epsilon(x, \cdot), \mu(x, \cdot)$ over the medium. They include the "correctors" which retain the microscopic interaction effects (multiple scattering) in the final approximation.

The correctors satisfy a system of the general form (see the following sections for details)

$$\nabla \cdot [\alpha(e_j + \nabla \chi_j^\alpha(y))] = 0, \quad \mathcal{M}[\chi_i] = 0 \quad (56)$$

where e_j is the j^{th} natural basis vector in \mathbb{R}^3 . In these equations \mathcal{M} is the operation of averaging over a typical cell in the domain for the periodic case; and it is expectation with respect to the stationary distribution describing the medium in the random case. When the conductivity $\sigma \neq 0$, the homogenized model is more complex, including a "memory effect" – see below.

3.4 Implementation of the Approximation

Thus, to implement the approximation, we must solve the equations (56) for the correctors and then compute the averages in (55). Computation of the correctors is possible in the periodic case (see, e.g., (Begis, Duvaut, Hassim 1981)(Bougat Derieux 1978)). However, in the random case of interest in this project, the computation is very difficult. The equations (56) in the random case are defined path by path; hence, one would have to have a complete statistical characterization of the medium to be able to solve them. This is impossible in practice, and one must evaluate the correctors by a procedure like Monte Carlo simulations. Hence, these equations should be regarded as the basis for derivation of further approximations or alternative representations for the medium which require only physically attainable statistical information, e.g., second order statistics in the microstructure of the medium. Such approximations have been derived in other problems (conductivity, porosity, etc.) in the past. We shall consider these methods and their adaptation to the EM scattering/absorption problem in the next section.

Before giving the details of the derivation of the approximation, we shall make a few remarks on its interpretation.

- First, note that the general form of the approximation is valid (with a very different mathematical interpretation) in both the random and periodic cases.

- Second, the role of the correctors in providing the "correct" form of the approximation cannot be dismissed. Approximations which simply average the variations of the permittivity and permeability over a domain are incorrect - making an error of "order one" in the small parameter ν . That is, the approximation (55)(56) is an approximation of "order ν " (at least in the periodic case); omitting the correctors invalidates this estimate.

3.5 Derivation of the Effective Parameters Using Homogenization

There are two basic cases to consider:

1. The periodic case when the inhomogeneities in the medium are spatially periodic; and
2. The case when the inhomogeneities in the medium are randomly distributed.

The expressions for the effective parameters are easier to grasp in the periodic case; and they are more readily computable. For this reason we shall treat this case first as background to the random case.

3.5.1 Periodic Media

Our treatment of this problem is based on (Sanchez-Palencia 1980).²

To begin, we rewrite Maxwell's equations

$$\frac{\partial D^\nu}{\partial t} = \nabla \times H^\nu - J^\nu + F \quad (57)$$

$$\frac{\partial B^\nu}{\partial t} = -\nabla \times E^\nu + G$$

$$E^\nu(x, 0) = 0, \quad H^\nu(x, 0) = 0$$

where F, G are (localized) source terms and³

$$D_i^\nu = \epsilon_{ij} \left(\frac{x}{\nu} \right) E_j^\nu, \quad B_i^\nu = \mu_{ij} \left(\frac{x}{\nu} \right) H_j^\nu, \quad J_i^\nu = \sigma_{ij} \left(\frac{x}{\nu} \right) E_j^\nu. \quad (58)$$

We assume that ϵ, μ, σ are smooth Y periodic functions of x/ν which are symmetric and positive definite.⁴ Adopting the multiscale hypothesis

$$E^\nu(x, t) = E^0(x, t) + \nu E^1(x, y, t) + \dots \quad (59)$$

$$H^\nu(x, t) = H^0(x, t) + \nu H^1(x, y, t) + \dots$$

and similarly for D^ν, B^ν and J^ν with each term in the asymptotic expansion periodic in y . We shall use the change of coordinates

$$\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x} + \frac{1}{\nu} \frac{\partial}{\partial y} \quad (60)$$

²See also (Bensoussan, Lions, Papanicolaou 1978) and (Tartar 1979).

³We use the *summation convention* throughout this section - repeated indices in an expression are summed.

⁴That is, $\epsilon_{ij} = \epsilon_{ji}$, $\epsilon_{ij} \xi_i \xi_j \geq \gamma |\xi|^2$, $\gamma > 0$.

with $y = x/\nu$, and similarly

$$(\nabla \times \cdot) \rightarrow (\nabla_z \times \cdot) + \frac{1}{\nu} (\nabla_y \times \cdot)$$

$$\text{div} \rightarrow \text{div}_z + \frac{1}{\nu} \text{div}_y.$$

Rewriting Maxwell's equations in terms of these coordinates, inserting the expansions (59) into them, and equating coefficients of like powers of ν , we arrive at a sequence of equations for $E^0, H^0, E^1, H^1, \dots$. The $O(\nu^{-1})$ and $O(\nu^0)$ terms are

$$\nabla_y \times H^0 = 0, \nabla_y \times E^0 = 0 \quad (61)$$

$$\frac{\partial D^0}{\partial t} = \nabla_z \times H^0 + \nabla_y \times H^1 - J^0 + F \quad (62)$$

$$\frac{\partial B^0}{\partial t} = -\nabla_z \times E^0 - \nabla_y \times E^1 + G$$

The equations (61) imply that $H_0(x, y, t), E_0(x, y, t)$ are gradients in y for fixed values of the other arguments.

The terms $\nabla_y \times E^1$ and $\nabla_y \times H^1$ are derivatives of periodic (in y) functions; hence, they have zero mean with respect to the averaging operation

$$\mathcal{M}(\nabla_y \times E^1) \triangleq \frac{1}{|Y|} \int_Y \nabla_y \times E^1 dy = 0 \quad (63)$$

(and similarly for $\nabla_y \times H^1$) where Y is a typical "cell" of the periodic structure and $|Y|$ is the volume of the cell. Therefore, averaging the system (62) over the cell Y , we arrive at the *homogenized system*

$$\frac{\partial \bar{D}^0}{\partial t} = \nabla_z \times \bar{H}^0 - \bar{J}^0 + F \quad (64)$$

$$\frac{\partial \bar{B}^0}{\partial t} = -\nabla_* \times \bar{E}^0 + G$$

where the overbar denotes the averaging operation. To complete the model we must derive the homogenized constituent equations corresponding to (58). These laws take two different forms, depending on the presence or absence of dissipation ($\sigma = 0$ or not).

Consider the divergence of (57)

$$\operatorname{div} \left(\frac{\partial D^\nu}{\partial t} + J^\nu \right) = \operatorname{div} F \quad (65)$$

$$\operatorname{div} \left(\frac{\partial B^\nu}{\partial t} \right) = \operatorname{div} G$$

Using the *ansatz* (59) and the coordinates (60), we have, at order ν^{-1} ,

$$\operatorname{div}_\nu \left(\frac{\partial D^0}{\partial t} + J^0 \right) = 0 \quad (66)$$

and

$$\operatorname{div}_\nu \left(\frac{\partial B^0}{\partial t} \right) = 0 \quad (67)$$

which, with the zero initial conditions, implies

$$\operatorname{div}_\nu B^0 = \operatorname{div}_\nu [\mu(x, y) H^0(x, y)] = 0 \quad (68)$$

From (61) we have that E^0 and H^0 are gradients; hence, we can write

$$E^0 - \bar{E}^0 = \nabla \phi, \quad H^0 - \bar{H}^0 = \nabla \psi \quad (69)$$

with the means of ϕ and ψ zero. Using these in (66)(67), we have

$$\frac{\partial}{\partial y_i} \left\{ \mu_{ij}(y) \left[\frac{\partial \psi}{\partial y_j} + \bar{H}_j^0 \right] \right\} = 0 \quad (70)$$

$$\frac{\partial}{\partial y_i} \left\{ \left[\epsilon_{ij}(y) \frac{\partial}{\partial t} + \sigma_{ij}(y) \right] \left[\frac{\partial \phi}{\partial y_j} + \bar{E}_j^0 \right] \right\} = 0 \quad (71)$$

These are the key equations in application of the homogenization method to Maxwell's equations. If we regard (70) as an equation for the potential ψ , then, noting that \bar{H}^0 does not depend on y , we can solve (70) by assuming the "separation of variables"

$$\psi(x, y, t) = \chi_k^\mu(x, y, t) \bar{H}_k^0(x, t) \quad (72)$$

and regarding (70) as an equation for $\chi(y)$ as a function of y , treating x, t as fixed parameters. We call $\chi_k^\mu(x, y, t), k = 1, 2, \dots, d$ (d is the dimension of the space) the *correctors* associated with μ_{ij} . Using (70)(72) the equation for the k^{th} corrector is

$$-\frac{\partial}{\partial y_i} \left[\mu_{ij} \frac{\partial \chi_k^\mu}{\partial y_j} \right] = \frac{\partial \mu_{ik}}{\partial y_i}, \quad k = 1, 2, \dots, d. \quad (73)$$

This problem plays a key role in the homogenization method. We call it the *cell problem*. The assumption that $\mu_{ij}(x, y)$ is periodic in y , symmetric, and positive definite (as a matrix for each x, y), guarantees that (73) has a unique solution⁵ which has zero mean

$$\bar{\chi} = \frac{1}{|Y|} \int_Y \chi dy = 0$$

The solvability condition for (73) is that the right hand side have zero mean. This holds since μ_{ij} is periodic and smooth.

⁵ Among the set of functions periodic in y with first derivatives square integrable over a cell, i.e., periodic functions in $H_{loc}^1(\mathbb{R}^d)$. See (Sanches-Palencia 1980), pp. 51-54, or (Bensoussan, Lions, Papanicolaou 1978).

The homogenized form of the field equations may be obtained using the correctors. Writing out the $O(\nu^0)$ term in the expansion of (65), we have

$$\operatorname{div}_y B^1 + \operatorname{div}_z B^0 = \operatorname{div}_z G \quad (74)$$

Using \mathcal{M} to denote the averaging operation, the solvability condition for this equation is

$$\mathcal{M}\{\operatorname{div}_z B^0 - \operatorname{div}_z G\} = 0$$

which we rewrite as

$$\mathcal{M}\left[\operatorname{div}_z\{\mu(x, y)[\nabla_y \psi + \bar{H}^0(x)] - G\}\right] = 0$$

Using the representation (72) for the potential, (75) holds, if

$$\mathcal{M}\left[\mu(\nabla_y \chi^\nu + \underline{1}) \bar{H}_0 - G\right] = 0.$$

This provides the definition of the *homogenized* magnetic permeability, μ^h , as

$$\mu_{ik}^h = \mathcal{M}\left(\mu_{ij}(x, y)\left[\delta_{ik} + \frac{\partial \chi_k^\mu}{\partial y_j}\right]\right) \quad (76)$$

The effective permittivity and conductivity will depend on the frequency of the incident radiation; and the analysis must reflect this. Recall (66), rewritten as

$$\operatorname{div}\left\{\frac{\partial}{\partial t}\epsilon E_0 + \sigma E_0\right\} = 0 \quad (77)$$

$$\begin{aligned} &= \operatorname{div}\left\{\left[\frac{\partial}{\partial t}\epsilon + \sigma\right][\nabla_y \phi + \bar{E}_0]\right\} \\ &= \frac{\partial}{\partial t}\operatorname{div}_y[\epsilon \nabla_y \phi] + \operatorname{div}_y[\sigma \nabla_y \phi] \end{aligned} \quad (78)$$

$$+\operatorname{div}_y \left[\epsilon \frac{\partial}{\partial t} \bar{E}_0 \right] + \operatorname{div}_y [\sigma \bar{E}_0]$$

Thus, the local "cell" equation is an evolution (in t) equation.

To treat (78) we use a technique introduced in (Sanchez-Palencia Sanchez-Hubbert 1978). Let V_Y be the space of suitably smooth functions $\theta(y)$ which are periodic with zero mean, and let V_Y be endowed with the inner product

$$\langle \phi, \theta \rangle_{V_Y} = \int_Y \epsilon_{ij} \frac{\partial \phi}{\partial y_i} \frac{\partial \theta}{\partial y_j} dy$$

Using this, (78) becomes

$$\begin{aligned} \frac{\partial}{\partial t} \int_Y \left[\epsilon_{ij} \frac{\partial \phi}{\partial y_j} + \epsilon_{ij} (\bar{E}_0)_j \right] \frac{\partial \theta}{\partial y_i} dy \\ + \int_Y \sigma_{ij} \frac{\partial \phi}{\partial y_j} \frac{\partial \theta}{\partial y_i} dy + \int_Y \sigma_{ij} (\bar{E}_0)_j \frac{\partial \theta}{\partial y_i} dy = 0 \end{aligned} \quad (79)$$

Using the inner product notation, this may be written compactly as

$$\frac{\partial}{\partial t} \langle \phi + c_j^1 (\bar{E}_0)_j, \theta \rangle_{V_Y} + \langle A\phi, \theta \rangle_{V_Y} + \langle c_j^2 (\bar{E}_0)_j, \theta \rangle_{V_Y} = 0 \quad (80)$$

where

$$\begin{aligned} \langle c_j^1, \theta \rangle_{V_Y} &= \int_Y \epsilon_{ij} \frac{\partial \theta}{\partial y_i} dy \\ \langle c_j^2, \theta \rangle_{V_Y} &= \int_Y \sigma_{ij} \frac{\partial \theta}{\partial y_i} dy \end{aligned}$$

The evolution equation (81) can be solved for the potential ϕ

$$\begin{aligned} \phi(t) &= -c_j^1 (\bar{E}_0)_j(t) + \int_0^t e^{-A(t-\tau)} c_j^3 (\bar{E}_0)_j(\tau) d\tau \\ c_j^3 &= Ac_j^1 - c_j^2 \end{aligned} \quad (81)$$

Using this expression for ϕ , we can compute the average flux vector \bar{D}_0 and average induced current \bar{J}_0 and define the effective permittivity and conductivity. Using $E_0 = \bar{E}_0 + \nabla\phi$, we write the average of $D_0 = \epsilon E_0$ as

$$\begin{aligned} (\bar{D}_0)_i &= \mathcal{M} \left\{ \epsilon_{ij} (\bar{E}_0 + \nabla\phi) \right\} \\ &= \mathcal{M} \left\{ \epsilon_{ik} (\bar{E}_0)_k \right\} \\ &+ \mathcal{M} \left\{ \epsilon_{ij} \left[\frac{\partial}{\partial y_k} \left(-c_j^1 (\bar{E}_0)_i(t) + \int_0^t e^{-\Lambda(t-\tau)} c_j^3 (\bar{E}_0)_i(\tau) d\tau \right) \right] \right\} \end{aligned} \quad (82)$$

This may be written compactly as

$$(\bar{D}_0)_i = a_{ij}^{\epsilon} (\bar{E}_0)_j(t) + \int_0^t b_{ij}^{\epsilon}(t-\tau) (\bar{E}_0)_j(t-\tau) d\tau \quad (83)$$

where

$$a_{ij}^{\epsilon} = \mathcal{M} \left[\epsilon_{ij}(y) - \epsilon_{ik}(y) \frac{\partial c_j^1(y)}{\partial y_k} \right] \quad (84)$$

$$b_{ij}^{\epsilon} = \mathcal{M} \left[\epsilon_{ij}(y) \frac{\partial}{\partial y_i} \left(e^{-\Lambda\tau} c_j^3 \right) (y) \right] \quad (85)$$

In a similar fashion, we can compute the induced current

$$(\bar{J}_0)_i = a_{ij}^{\sigma} (\bar{E}_0)_j(t) + \int_0^t b_{ij}^{\sigma}(t-\tau) (\bar{E}_0)_j(t-\tau) d\tau \quad (86)$$

where a^{σ} and b^{σ} are defined by (85)(86) with ϵ replaced by σ .

Using Laplace transforms, we can more readily express the effective scattering parameters as a function of frequency. Indeed, if we let $\hat{D}^{\nu}(\omega)$ be the Laplace transform of D^{ν} with respect to t , and similarly for the other variables, then Maxwell's equations take the form

$$\omega \hat{D}^{\nu} + \hat{J}^{\nu} = \nabla \times \hat{H}^{\nu} + \hat{F} \quad (87)$$

$$\omega \hat{B}^\nu = -\nabla \times \hat{E}^\nu + \hat{G}$$

The claim is that Maxwell's equations approach the homogenized laws

$$\omega \hat{D}^h + \hat{J}^h = (\omega \epsilon + \sigma)^h \hat{E}^h \quad (88)$$

$$\hat{B}^h = \mu^h \hat{H}^h$$

where the superscript $(\cdot)^h$ denotes the "homogenized" laws. Using the evolution equations (83)(86) we can identify

$$\omega a_{ij}^\epsilon + \omega \hat{b}_{ij}^\epsilon(\omega) + a_{ij}^\sigma + \hat{b}_{ij}^\sigma(\omega) = (\omega \epsilon_{ij} + \sigma_{ij})^h, \quad \Re \omega > 0 \quad (89)$$

This completes the derivation of the (first order) homogenization theory for Maxwell's equations.

Remark: Using the arguments in (Bensoussan, Lions, Papanicolaou 1978) and (Sanchez-Palencia Sanchez-Hubbert 1978) and (Sanchez-Palencia 1980), it can be shown that the homogenized Maxwell's equations (83)(86) (or the frequency domain version) have a unique solution. Moreover, it can be shown that the homogenized Maxwell's equations are the limits as $\nu \rightarrow 0$ of the original scaled Maxwell's equations (in an appropriate weak* topology).

Rather than give these arguments, we shall turn our attention to the case when the coefficients in Maxwell's equations are random processes in the spatial variables.

3.5.2 Random Media

As we have noted earlier, the formulas for the effective parameters in the homogenized form of Maxwell's equations carry over in form, at least, to the random case. The averaging operation M is interpreted as expectation with respect to an appropriate probability measure. While the procedure of simply expanding the field quantities in asymptotic series in the small parameter is only formal in the random case, it can still be used with appropriate cautions to determine equations for the homogenized field quantities. The arguments for convergence of the scaled field quantities to the homogenized quantities are, however, substantially different in the random case.

The central difficulty in treating random coefficients arises in the meaning and evaluation of the "correctors" $\chi^e, \chi^\mu, \chi^\sigma$. They are random processes which must be evaluated (numerically) by a Monte Carlo type procedure. We shall take up this issue later in the project. In this section we shall examine the more fundamental issue of existence and uniqueness of solutions to the corrector equations. Since each of the three correctors $\chi^e, \chi^\mu, \chi^\sigma$ satisfies the same type of elliptic equation with random coefficients, we shall treat the generic problem: Find $\chi(y; \omega)$ such that

$$\chi(0; \omega) = 0, E\chi(y) = 0 \quad \forall y \quad (90)$$

$\partial\chi/\partial y_j$ is a square integrable stationary process $\forall j$ and

$$-\frac{\partial}{\partial y_i} \left(a_{ij}(y, \omega) \frac{\partial\chi}{\partial y_j} \right) = \frac{\partial g_j}{\partial y_j}(y, \omega)$$

in the sense of distributions. We assume that

$$a_0 |\xi|^2 \leq \sum_{i,j} a_{ij}(y, \omega) \xi_i \xi_j \leq \frac{1}{a_0} |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, a_0 > 0 \quad (91)$$

where $a_{ij}(y, \omega)$ and $g_j(y, \omega)$ are square integrable stationary processes, $i, j = 1, \dots, n$.

Notation - Assumptions We shall consider a set up for stationary processes as presented in Papanicolaou and Varadhan (Papanicolaou Varadhan 1979). Let (Ω, \mathcal{A}, P) be a probability space and define $\mathcal{H} = L^2(\Omega, \mathcal{A}, P)$. We assume that

$$\mathcal{H} \text{ is separable} \quad (92)$$

$$\text{there exists a strongly continuous unitary group } T_y \text{ on } \mathcal{H}, y \in \mathbb{R}^n \quad (93)$$

$$T_y \text{ is ergodic, which means if } \tilde{f} \in \mathcal{H} \text{ satisfies} \quad (94)$$

$$T_y \tilde{f} = \tilde{f}, \quad \forall y, \text{ then } \tilde{f} \text{ is a constant.}$$

$$\text{if } \tilde{f} \geq 0, \text{ then } T_y \tilde{f} \geq 0 \text{ and } T_y 1 = 1 \quad (95)$$

The group T_y has a spectral resolution defined by

$$T_y = \int_{\mathbb{R}^n} e^{i\lambda y} U(d\lambda)$$

where $U(d\lambda)$ is a projection valued measure. We consider the complex extension of \mathcal{H} , provided with the scalar product $E\tilde{f}\tilde{g}$, whenever \tilde{f}, \tilde{g} are two elements of \mathcal{H} . The measure $U(\Delta)$, Δ Borel subset of \mathbb{R}^n , satisfies

$$EU(\Delta)\tilde{f}\overline{U(\Delta')\tilde{g}} = 0 \quad \forall \tilde{f}, \tilde{g} \in H, \Delta \cap \Delta' = \emptyset$$

$$EU(\Delta)\tilde{f}U(\bar{\Delta})\tilde{g} = EU(\Delta)\tilde{f}\tilde{g}$$

and by ergodicity

$$U(\{0\})\tilde{f} = E\tilde{f} \quad (96)$$

We next define

$$D_i\tilde{f}(\omega) = \frac{\partial}{\partial y_i} (T_v\tilde{f})(\omega)|_{v=0} \quad (97)$$

which are closed densely defined linear operators with domains $\mathcal{D}(D_i)$ in \mathcal{H} .

Note that

$$E\{\tilde{g}D_i\tilde{f}\} = -E\{D_i\tilde{g}\tilde{f}\} \quad \forall \tilde{f}, \tilde{g} \in \mathcal{D}(D_i), \quad (98)$$

and

$$D_j\tilde{f} = i \int_{\mathbb{R}^n} \lambda_j U(d\lambda)\tilde{f}. \quad (99)$$

If $D_j\tilde{f} = 0 \quad \forall j$, then since

$$E|D_j\tilde{f}|^2 = \int_{\mathbb{R}^n} \lambda_j^2 E|U(d\lambda)\tilde{f}|^2$$

it follows that $U(d\lambda)\tilde{f} = 0 \quad \forall \lambda \neq 0$, hence

$$\tilde{f} = U(\{0\})\tilde{f} = E\tilde{f}.$$

Let $\mathcal{H}^1 = \bigcap_{j=1}^n \mathcal{D}(D_j)$ which is dense in \mathcal{H} . We equip \mathcal{H}^1 with the Hilbert scalar product

$$((\tilde{f}, \tilde{g}))_{\mathcal{H}^1} = E\tilde{f}\tilde{g} + \sum_{j=1}^d ED_j\tilde{f}D_j\tilde{g} \quad (100)$$

We identify \mathcal{H} with its dual and call \mathcal{H}^{-1} the dual of \mathcal{H}^1 . We have the inclusions

$$\mathcal{H}^1 \subset \mathcal{H} \subset \mathcal{H}^{-1}$$

each space being dense in the next one with continuous injection.

The family T_v is also a strongly continuous unitary group on \mathcal{H}^1 , since

$$T_v D_i \tilde{f} = D_i T_v \tilde{f}.$$

It can be extended as a strongly continuous unitary group on \mathcal{H}^{-1} by the formula

$$\langle T_v \tilde{f}_\bullet, \tilde{f} \rangle = \langle \tilde{f}_\bullet, T_{-v} \tilde{f} \rangle \quad \forall \tilde{f} \in \mathcal{H}^1, \tilde{f}_\bullet \in \mathcal{H}^{-1}$$

and \langle, \rangle refers to the duality between \mathcal{H}^1 and \mathcal{H}^{-1} .

Remark 1. *The periodic case.*

Let Ω be the unit n dimensional torus, \mathcal{A} the σ -algebra of Lebesgue measurable sets and P Lebesgue measure on Ω . Then \mathcal{H} is the space of measurable periodic functions (period 1 in each component) such that

$$\int_{\Omega} (\tilde{f}(\omega))^2 d\omega < \infty.$$

We define

$$T_v \tilde{f}(\omega) = \tilde{f}(\omega + y)$$

hence

$$D_i \tilde{f} = \frac{\partial}{\partial \omega_i} \tilde{f}.$$

An important fact in the periodic case which does not carry over to the random case, is that there is no analogue of Poincare's inequality. The consequence is following. Consider the quotient space $\mathcal{H}^1/\mathfrak{R}$ of elements of \mathcal{H}^1 which differ by a constant. Denote by $[\tilde{f}]$ the equivalence class related to an element \tilde{f} , then the quotient norm is given by

$$\| [\tilde{f}] \| = \{ E|\tilde{f} - E\tilde{f}|^2 + \sum_j E|D_j \tilde{f}|^2 \}^{1/2}$$

This is not equivalent to $(\sum_j E|D_j \tilde{f}|^2)^{1/2}$. In the periodic case one has

$$\| [\tilde{f}] \|^2 = \int_{\Omega} |\tilde{f} - m(\tilde{f})|^2 d\omega + \sum_j \int_{\Omega} \left| \frac{\partial \tilde{f}}{\partial \omega_j} \right|^2 d\omega$$

where $m(\tilde{f}) = \int_{\Omega} \tilde{f}(\omega) d\omega$. Poincare's inequality implies that $\| [\tilde{f}] \|$ is equivalent to $(\sum_j \int_{\Omega} \left| \frac{\partial \tilde{f}}{\partial \omega_j} \right|^2 d\omega)^{1/2}$.

Consider now random variables not necessarily in \mathcal{H} . We assume that

T_y is a linear group on the set of complex random variables such that $\forall \tilde{\eta}_1 \dots \tilde{\eta}_k$ complex random variables, ψ Borel bounded function on C^k , then

$$E\phi(T_y \tilde{\eta}_1, \dots, T_y \tilde{\eta}_k) = E\phi(\tilde{\eta}_1, \dots, \tilde{\eta}_k) \quad (101)$$

$y, \omega \rightarrow T_y \tilde{\eta}$ is measurable,

$$T_y \tilde{\eta} \geq 0 \text{ if } \tilde{\eta} \geq 0.$$

A *stationary process* is a stochastic process $\eta(y; \omega)$ which can be represented in the form

$$\eta(y; \omega) = T_y \tilde{\eta}(\omega). \quad (102)$$

The space of square integrable stationary processes can be identified with \mathcal{H} . Moreover

$$\begin{aligned} \frac{\partial \eta}{\partial y_i}(y; \omega) &= D_i T_y \tilde{\eta}(\omega) = T_y D_i \tilde{\eta} \\ &\text{if } \tilde{\eta} \in \mathcal{H}^1. \end{aligned} \quad (103)$$

Note that the continuity assumption on T_y implies that the square integrable stationary processes are necessarily continuous function of y with values in \mathcal{H} . Hence, if $\tilde{\eta} \in \mathcal{H}$, $\eta(y; \omega) \in C^0(\mathbb{R}^n; \mathcal{H})$, space of uniformly continuous functions on \mathbb{R}^n with values in \mathcal{H} . If $\tilde{\eta} \in \mathcal{H}^1$ then $\eta(y; \omega) \in C^1(\mathbb{R}^n; \mathcal{H})$.

Note that $D_i \in \mathcal{L}(\mathcal{H}^1; \mathcal{H})$. If $\tilde{\eta} \in \mathcal{H}$, we can consider the distribution derivative $\partial \eta / \partial y_i$ with values in \mathcal{H} , defined by

$$\int_{\mathbb{R}^n} \frac{\partial \eta}{\partial y_i} \theta(y) dy = - \int_{\mathbb{R}^n} \eta(y; \omega) \frac{\partial \theta}{\partial y_i} dy, \quad \forall \theta \in C_0^\infty(\mathbb{R}^n).$$

Let us check that

$$- \int_{\mathbb{R}^n} \eta(y; \omega) \frac{\partial \theta}{\partial y_i} dy = \int_{\mathbb{R}^n} D_i T_y \tilde{\eta} \theta(y) dy \quad (104)$$

which is an equality in \mathcal{H}^{-1} . This proves that $\eta(y; \omega) \in C^{-1}(\mathbb{R}^n; \mathcal{H}^{-1})$ and the distribution derivative $\partial \eta / \partial y_i$ with values in \mathcal{H} can be considered as a continuous function with values in \mathcal{H}^{-1} .

To prove (104) pick $\tilde{\phi} \in \mathcal{H}^1$, then

$$\begin{aligned}
E \tilde{\phi} \int_{\mathbb{R}^n} D_i T_y \tilde{\eta} \theta(y) dy &= -E \int_{\mathbb{R}^n} D_i \tilde{\phi} T_y \tilde{\eta} \theta(y) dy \\
&= -E \int_{\mathbb{R}^n} D_i T_{-y} \tilde{\eta} \theta(y) dy \\
&= -E \int_{\mathbb{R}^n} \frac{\partial \phi}{\partial y}(-y, \omega) \tilde{\eta} \theta(y) dy \\
&= -E \int_{\mathbb{R}^n} \phi(-y, \omega) \tilde{\eta} \frac{\partial \theta}{\partial y_i}(y) dy \\
&= -E \int_{\mathbb{R}^n} \tilde{\phi} \eta(y, \omega) \frac{\partial \theta}{\partial y}(y) dy
\end{aligned}$$

which completes the proof of (104).

The Cell Problem We consider here stationary processes $a_{ij}(y; \omega)$ such that

$$a_0 |\xi|^2 \leq \sum_{i,j} a_{ij}(y, \omega) \xi_i \xi_j \leq \frac{1}{a_0} |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, a_0 > 0 \quad (105)$$

Let $g_j(y, \omega) = T_y \tilde{g}_j$ square integrable stationary processes, $j = 1, \dots, n$. We shall solve the problem: Find

$$\chi(y; \omega) \in C^1(\mathbb{R}^n; \mathcal{H}), \chi(0; \omega) = 0, E\chi(y) = 0 \quad \forall y \quad (106)$$

$\partial \chi / \partial y_j$ is a square integrable stationary process $\forall j$

$$-\frac{\partial}{\partial y_i} \left(a_{ij}(y, \omega) \frac{\partial \chi}{\partial y_j} \right) = \frac{\partial g_j}{\partial y_j}(y, \omega)$$

in the sense of distributions with values in \mathcal{H} (or as continuous functions with values in \mathcal{H}^{-1}).

Papanicolaou and Varadhan (Papanicolaou Varadhan 1979) have shown the existence and uniqueness of the solution of (106). We shall reproduce their proof with minor changes. Note that $\chi(y; \omega)$ itself is *not* a stationary process. This is a big difference with respect to the periodic case and relates to Remark 1. Note also that $\chi(y; \omega) \in C^2(\mathbb{R}^n; \mathcal{H}^{-1})$.⁶

Let $\tilde{\chi}_j \in H$ such that

$$\frac{\partial \chi}{\partial y_j}(y; \omega) = T_v \tilde{\chi}_j$$

we can assert that

$$E \tilde{\chi}_j D_k \tilde{\phi} = E \tilde{\chi}_k D_j \tilde{\phi}, \quad \forall \tilde{\phi} \in \mathcal{H}^1. \quad (107)$$

Indeed we have to show that

$$D_k \tilde{\chi}_j = D_j \tilde{\chi}_k \quad (108)$$

as an equality in \mathcal{H}^{-1} . But

$$\begin{aligned} T_v D_k \tilde{\chi}_j &= \frac{\partial}{\partial y_k} \chi_j(y; \omega) \\ &= \frac{\partial^2 \chi}{\partial y_j \partial y_k}(y; \omega) \end{aligned}$$

hence

$$T_v D_k \tilde{\chi}_j = T_v D_j \tilde{\chi}_k$$

⁶By virtue of (95) it is sufficient to have (105) for $y = 0$.

which implies (108).

We have also

$$E\tilde{\chi} = 0. \quad (109)$$

This follows from

$$\chi(y; \omega) = \sum_j \int_0^1 T_{\theta_j} \tilde{\chi}_j y_j d\theta$$

hence

$$E(\chi/y) = \sum_j E\tilde{\chi}_j y_j = 0$$

by the assumption. Therefore (109) follows.

We can then state

Theorem 1. *There exists one and only one solution of (106).*

Proof.

Uniqueness.

Assume that $\partial g_j / \partial y_j = 0$. Define

$$\tilde{\phi}^{\beta}(\omega) = \int_{\mathbb{R}^n} \sum_j \frac{(-i\lambda_j - \beta)}{|i\lambda - \beta|^2} U(d\lambda) \tilde{\chi}_j(\omega). \quad (110)$$

Note that (106) can be written as

$$-D_i T_{\nu}(\tilde{a}_{i\nu} \tilde{\chi}_j) = 0 \text{ in } \mathcal{H}^{-1}$$

hence, as is easily seen

$$E\bar{a}_{ij}\tilde{\chi}_j D_i \tilde{\phi}^\beta = 0. \quad (111)$$

Because of (108) we have

$$\int_{\mathbb{R}^n} \frac{\lambda_k U(d\lambda) \tilde{\chi}_j}{|i\lambda - \beta|^2} = \int_{\mathbb{R}^n} \frac{\lambda_j U(d\lambda) \tilde{\chi}_k}{|i\lambda - \beta|^2}$$

which implies

$$\begin{aligned} D_k \tilde{\phi}^\beta - \beta \tilde{\phi}^\beta &= \int_{\mathbb{R}^n} \sum_j \frac{(-i\lambda_j - \beta)(i\lambda_k - \beta)}{|i\lambda - \beta|^2} U(d\lambda) \tilde{\chi}_j(\omega) \\ &= \int_{\mathbb{R}^n} \sum_j \frac{(-i\lambda_j - \beta)(i\lambda_j - \beta)}{|i\lambda - \beta|^2} U(d\lambda) \tilde{\chi}_k(\omega) = \tilde{\chi}_k. \end{aligned}$$

Therefore, (111) reads

$$E\bar{a}_{ij}\tilde{\chi}_j\tilde{\chi}_i + \beta E\bar{a}_{ij}\tilde{\chi}_j\tilde{\phi}^\beta = 0. \quad (112)$$

However,

$$\begin{aligned} E|\beta \tilde{\phi}^\beta|^2 &= \int_{\mathbb{R}^n} \beta^2 \sum_{j,k} \frac{(-i\lambda_j - \beta)(-i\lambda_k - \beta)}{|i\lambda - \beta|^4} EU(d\lambda) \tilde{\chi}_j \tilde{\chi}_k \\ &\rightarrow \sum_{j,k} EU\{0\} \tilde{\chi}_j \tilde{\chi}_k \text{ as } \beta \rightarrow 0 \end{aligned}$$

and by ergodicity and property (109), we get

$$E|\beta \tilde{\phi}^\beta|^2 \rightarrow 0 \text{ as } \beta \rightarrow 0.$$

Therefore, (112) implies

$$E\bar{a}_{ij}\tilde{\chi}_j\tilde{\chi}_i = 0$$

hence $\tilde{\chi}_i = 0$, which implies also $\chi = 0$.

Existence:

Let $\beta > 0$, we solve the problem

$$\begin{aligned} -\frac{\partial}{\partial y}(\alpha_{ij} \frac{\partial \chi^\beta}{\partial y_0}) + \beta \chi^\beta &= \frac{\partial g_j}{\partial y_j} \\ \chi^\beta(y; \omega) &= T_y \tilde{\chi}^\beta, \tilde{\chi}^\beta \in \mathcal{H}^1 \end{aligned} \quad (113)$$

This problem is equivalent to

$$E \bar{\alpha}_{ij} D_j \tilde{\chi}^\beta D_i \tilde{\phi} + \beta E \tilde{\chi}^\beta \tilde{\phi} = -E \bar{g}_j D_j \tilde{\phi}, \quad \forall \tilde{\phi} \in \mathcal{H}^1 \quad (114)$$

We easily deduce the estimates

$$E |D_j \tilde{\chi}^\beta|^2 \leq C$$

$$\beta E (\tilde{\chi}^\beta)^2 \leq C.$$

Let us extract a subsequence such that

$$D_j \tilde{\chi}^\beta \rightarrow \tilde{\chi}_j \text{ in } H \text{ weakly.}$$

$$E D_j \tilde{\chi}^\beta D_k \tilde{\phi} = E D_k \tilde{\chi}^\beta D_j \tilde{\phi}$$

$$E D_j \tilde{\chi}^\beta = 0$$

we deduce (108) and (109). Going to the limit in (114) we have

$$E \bar{\alpha}_{ij} \tilde{\chi}_j D_i \tilde{\phi} = -E \bar{g}_j D_j \tilde{\phi} \quad \forall \tilde{\phi} \in \mathcal{H}^1. \quad (115)$$

Define then

$$\chi(y; \omega) = \int_{\mathbb{R}^n} (e^{i\lambda y} - 1) \frac{1}{|\lambda|^2} \sum_j (-i\lambda_j) U(d\lambda) \tilde{\chi}_j(\omega) \quad (116)$$

then

$$\begin{aligned} \frac{\partial \chi}{\partial y_k}(y; \omega) &= \int_{\mathbb{R}^n} e^{i\lambda y} \frac{1}{|\lambda|^2} \sum_j \lambda_j \lambda_k U(d\lambda) \tilde{\chi}_j(\omega) \\ &= T_\nu \tilde{\chi}_k \end{aligned}$$

and $\chi(0; \omega) = 0$, $E\chi(y) = 0$. Then (115) can be written as

$$-D_i \tilde{a}_{ij} \tilde{\chi}_j = D_i \tilde{g}_i \text{ equality in } \mathcal{H}^{-1}$$

which is indeed (106).

4 Bounds for Effective Parameters

In previous sections we have examined methods for representation and computation of effective parameter models for heterogeneous media in terms of statistical characterizations of the media. These methods lead to representations for the effective parameters, e.g., the effective dielectric constant and conductivity, which are difficult to evaluate in any but the simplest cases. In this section we examine the derivation of *bounds* on the effective parameters as a means of quickly providing good approximations to the behavior of the medium.

There has been a great deal of work in this area, not only in electromagnetics, but also in conductivity and elasticity. This work dates from the

early years of this century (Bergman 1978), and it includes some very recent work (e.g., (Ericksen, et al. 1986)). We have been studying the portions of this work applicable to the case at hand - scattering and absorption of EM radiation by foliage covered terrain. Several key issues must be resolved before the work can be applied directly to scattering/absorption from foliage with more than one kind of elementary scatterer. We shall point out these issues in the following paragraphs.

4.1 Definition of the Effective Parameters

The starting point is a definition of the effective parameter, e.g., the effective permittivity. Suppose

$$\epsilon(x, \omega) = \epsilon_1 \mathbf{1}_1(\omega) + \cdots + \epsilon_N \mathbf{1}_N(\omega) \quad (117)$$

is the dielectric "constant" of a composite medium, where $\epsilon_i, i = 1, \dots, N$ is the (complex) dielectric constant of medium i , and the indicator function $\mathbf{1}_i$ is 1 for all samples $\omega \in \Omega$ which have medium i at point x , and zero otherwise. We use $(\Omega, \mathcal{F}, \mathcal{P})$ to denote the probability space on which the statistics of the phenomena are defined. The quantity

$$f_j = \int_{\Omega} P(d\omega) \mathbf{1}_j(\omega) \quad (118)$$

is the *volume fraction* occupied by the material of type $j = 1, 2, \dots, N$.

The spatially dependent random conductivity and magnetic permeability are similarly defined. In our case these functions depend on the random

geometry of the scattering foliage which may be available only in terms of second order statistics. The bounds obtained for the "effective parameter" approximations of these functions should not depend on statistics beyond this level.

If $E(x, \omega)$ and $D(x, \omega)$ are the electric and displacement field vectors (stationary in time), then

$$D_i(x, \omega) = \epsilon E_i(x, \omega) \quad (119)$$

$$\nabla \cdot D(x, \omega) = 0$$

$$\nabla \times E(x, \omega) = 0$$

and

$$\int_{\Omega} P(d\omega) E(x, \omega) = \bar{E}$$

provides the boundary condition, where \bar{E} is a constant field incident on the region. By normalizing the field magnitudes and redefining the coordinates, we can take $\bar{E} = e_k$, the k^{th} unit vector in \mathbb{R}^3 . Let E^k and D^k denote the corresponding fields. Then (119) may be rewritten

$$D_i^k(x, \omega) = \sum_{j=1}^d \epsilon_{ij}(x, \omega) E_j^k(x, \omega) \quad (120)$$

The *effective dielectric constant* may be defined as

$$\epsilon_{ik}^* = \int_{\Omega} P(d\omega) D_i^k(\omega) \quad (121)$$

In (Golden Papanicolaou 1983) it was shown that this ensemble average coincides with the usual definition involving a volume average.

Equation (121) may also be written in the symmetric form

$$\epsilon_{ik}^* = \int_{\Omega} P(d\omega) \sum_{j=1}^d E_j^k \bar{E}_j^i \quad (122)$$

where the bar denotes complex conjugate. This expression suggests the interpretation of ϵ^* as the dielectric constant of a fictitious, homogeneous medium (for the volume V) which provides the same value for the electrostatic energy stored in the volume occupied by the heterogeneous medium.⁷

4.2 Methods for Computing Bounds

Approximation to the effective permittivity (or conductivity in thermal systems) have been derived by three basic methods. The original work of Haskin and Shtrikman (1962) was based on variational principles, and these have been reformulated in recent years to provide one of the basic methods for deriving approximations (Milton 1981a, 1981b). The second method for deriving bounds is the method of "compensated compactness" (Tartar 1986). The third method involves the use of complex analysis and "representation formulas" (Bergman 1978)(Golden 1986)(Golden Papanicolaou 1983). We shall review the first two methods in detail in another report. Here we shall briefly describe the analytical approach to deriving bounds, indicate its range of applicability, and discuss our ideas for adapting the

⁷The volume average form is $\epsilon_{ik}^* |E_0|^2 = [\int_V \epsilon(r) |E(r)|^2 dV] / \text{meas}(V)$ where V is the representative region, $\text{meas}(V)$ is its volume, and $E_0 = [\int_V E(r) dV] / \text{meas}(V)$ is the average (incident) field.

method to treat scattering and absorption from foliage.

If we substitute (117) into (121) and divide by ϵ_N , then we obtain

$$m_{ik}(h_1, \dots, h_{N-1}) = \frac{\epsilon_{ik}^*}{\epsilon_N} = \int_{\Omega} P(d\omega) \left(\sum_{j=1}^{N-1} h_j 1_j(\omega) + 1_N \right) E_i^k(\omega) \quad (123)$$

Here $h_i = \epsilon_i / \epsilon_N$. The key idea in the work of Bergman (Bergman 1978) is to regard $m(\cdot)$ as a function of the complex variables h_1, \dots, h_{N-1} in \mathbb{C}^{N-1} . This permits one to use (spectral) representation formulas for $m(\cdot)$ which allow the derivation of bounds based on the extreme points of certain sets of measures. By expanding the representation formulas about the case of a "homogeneous" medium $h_j = 1, j = 1, \dots, N-1$, it is possible to obtain a characterization of the underlying measures in terms of their "moments." These moments are functions of increasing amounts of information on the statistical properties of the medium. For example, if only the volume fractions f_j from (118) are known, then only the first term in the representation can be computed explicitly. This leads to the classical Wiener bounds:

For two-component media -

$$\left(\frac{f_1}{\epsilon_1} + \frac{f_2}{\epsilon_2} \right)^{-1} \leq \epsilon^* \leq f_1 \epsilon_1 + f_2 \epsilon_2 \quad (124)$$

For three-component media -

$$\left(\frac{f_1}{\epsilon_1} + \frac{f_2}{\epsilon_2} + \frac{f_3}{\epsilon_3} \right)^{-1} \leq \epsilon^* \leq f_1 \epsilon_1 + f_2 \epsilon_2 + f_3 \epsilon_3 \quad (125)$$

These bounds are achieved by parallel plane configurations of the materials.

If additional information is available on the statistical description of the medium, then more sophisticated bounds can be obtained. For example, for a two-component medium which is statistically isotropic the first moment of the spectral measure can be computed (in the expansion). This leads to the Hashin-Shtrikman bounds (assume without loss of generality $\epsilon_1 \leq \epsilon_2$)

$$\epsilon_1 + \frac{f_2}{1/(\epsilon_2 - \epsilon_1) + f_1/3\epsilon_1} \leq \epsilon^* \leq \epsilon_2 + \frac{f_1}{1/(\epsilon_1 - \epsilon_2) + f_2/3\epsilon_2} \quad (126)$$

4.3 Multicomponent Media

There is a substantial difficulty in passing from two-component materials to N -component materials with $N \geq 3$ when using the analytical continuation method. Basically, one must deal with functions of more than one complex variable (h_1, h_2, \dots). The precise difficulty arises from the fact that the extreme points of the associated set of spectral measures are not known. Prior to the paper (Golden, Papanicolaou 1985) there was no systematic method for treatment of multi-component media (using the continuation method). In the case of three-component media, it is possible to circumvent this problem by imposing a linear relationship among the complex parameters, and expanding the spectral measure about the (linear) parameter. In (Golden, Papanicolaou 1985) an unconfirmed technical hypothesis was introduced which permitted recovery of the Hashin-Shtrikman bounds for three-component media. Assuming $\epsilon_1 \leq \epsilon_2 \leq \epsilon_3$ we have

$$\epsilon_1 + 1/\left(\frac{1}{A_1} - \frac{1}{3\epsilon_1}\right) \leq \epsilon^* \leq \epsilon_3 + 1/\left(\frac{1}{A_3} - \frac{1}{3\epsilon_3}\right) \quad (127)$$

$$A_j = \sum_{i=1} 3f_i / \left(\frac{1}{\epsilon_i - \epsilon_j} + \frac{1}{3\epsilon_j} \right)$$

These bounds are "optimal" for materials consisting a mixture of spheres of all different sizes of ϵ_1 and ϵ_2 materials each coated with ϵ_3 material in the appropriate volume fraction. Additional bounds were obtained in (Golden 1986) for complex permittivities (thus including the absorption of energy in the material).

The analytical continuation method is a powerful technique for derivation of bounds on the material properties of multi-component media; and it is useful to examine its potential for the evaluation of scattering/absorption phenomena in the interaction of EM radiation and foliage. Our evaluation of this methodology is not yet complete; however, the following points are worth making at this time:

- The method is based on the assumption that the medium is nearly homogeneous; that is, the normalized permittivities $\epsilon_j/\epsilon_N \approx 1$. This is a restrictive assumption for foliage, and it may limit application of the approximation to certain kinds of foliage (mixtures of grasses, etc.).
- While the case of two component media is reasonably well developed; the case of three-component media is less clear. This is the key case if one is to understand the interaction of scattering effects in an environment of two different types of scatterers in a uniform background. The limitations on the method are both technical and physical.

- The technical limitations are mainly due to the incomplete characterization of the set of spectral measures for functions of two complex variables.
- The physical limitations are more fundamental. The method does not account for the geometry of multiple scattering processes. That is, while it is capable, in principle, of including higher order statistical functions in the expansion for the effective permittivity, the method develops the bounds in a fashion which is well removed from the physical properties of the medium. The extreme points of the set of measures which are used to determine the bounds are singular measures ("delta functions") which may not correspond well to the distributions of scatterers in typical foliage. It would be better to develop the bounds in a (restricted) set of measures which are representative of foliage. It appears to be possible to adapt the analytical continuation method to this case.

We are continuing our investigation of this methodology for the construction of bounds with an eye to improving on these limitations. Our objective is to compare the approximations produced by this method with those derived through the other techniques we have studied and develop a hierarchy of approximations for scattering in foliage (of various types). We shall design a series of numerical experiments to test the methods on representative data.

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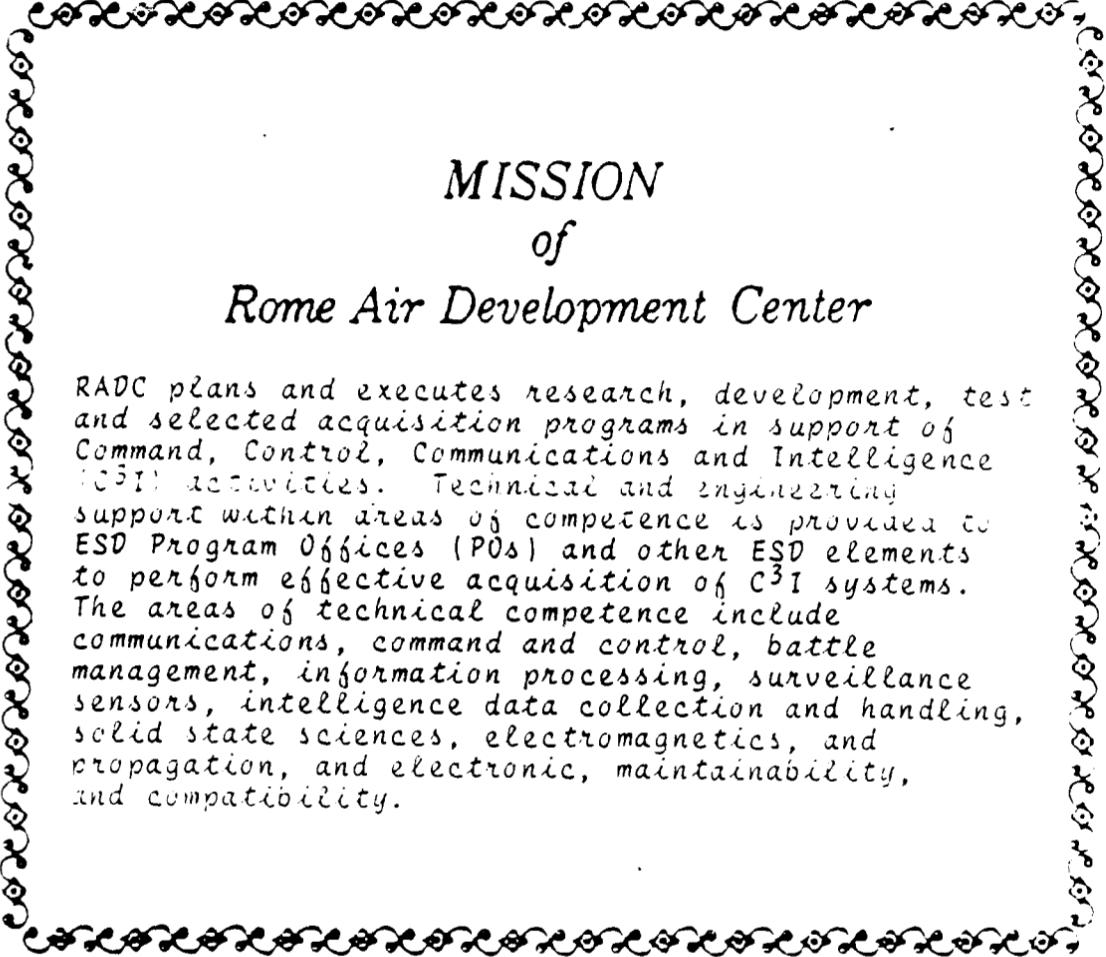
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